OVERVIEW OF SPREADING + NON-LOCALITY

PHYS 235 - NL PLASMA THEORY

THIS SECTION OF NOTES PREPARED BY

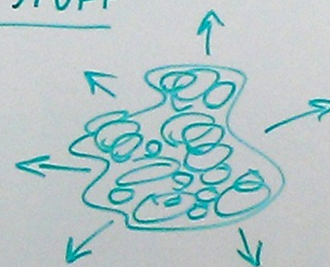
$\boxed{\text{NOAH HURST}}$ UCSD PHYSICS

WINTER 2014

INTRO / OUTLINE

I. LOITSYANSKY INTEGRAL + RELATED STUFF

→ Given bounded "patch" of turbulence
How, why does it spread??



II. AVALANCHING

→ Could this process (Familiar from snow/rock physics) explain "Non-Local" transport in tokamaks?

III. K - ε TURBULENCE MODELS

→ Can we accurately describe turbulent flow + associated phenomena by introducing a small-scale cutoff?

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYASKY INTEGRAL

Take Kolmogorov theory (1941) one step further

⇓

- ✓ homogeneous
- ✓ isotropic
- ✓ effectively infinite (i.e. local ...)
- ✓ steady-state

* S.S. energy input via stirring @ outer scale l_0

* S.S. energy cascade in k-space, terminating at inner scale l_d

⇓

- ✓ homogeneous
- ✓? isotropic → briefly discuss anisotropy
- ✗ infinite
- ✗ S.S. → consider bounded "patch" of turbulence

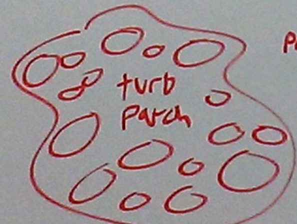
↓

given initial patch at $t=0$, how does it freely evolve? (stirring absent)

LARGE-SCALE END OF K41

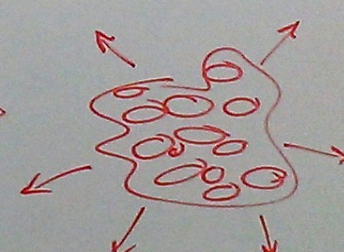
→ transport, mixing, spreading, etc ...

$t=0$



Fluid outside patch boundary at rest out to ∞ ⇒

$t>0$



SPREADING

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYASKY INTEGRAL

TIMELINE

- 1938 KARMAN-HOWARTH EQN
- 1939 LOITSYANSKY "Some basic laws of isotropic turbulent flow"
  - → Loitsyansky integral I is conserved, $\frac{d}{dt} I = 0$
- 1941 KOLMOGOROV THEORY → uses $\frac{d}{dt} I = 0$
- ? LANDAU arguments somewhere in here → Also claim $\frac{d}{dt} I = 0$
- 1954 PROUDMAN & REID "On the decay of a normally distributed and homogeneous turbulent velocity field"
  - → suggest that maybe $\frac{d}{dt} I \neq 0$
- 1956 BATCHELOR & PROUDMAN "The large-scale structure of homogeneous turbulence"
  - → Show that $\frac{d}{dt} I \neq 0$ under certain conditions
- 1966 SAFFMAN ... same title as above
  - → show that I diverges (sometimes), propose different invariant
- 2000 DAVIDSON "The role of angular momentum in isotropic turbulence"
  - → under certain (typical?) conditions, turb reaches asymptotic state where $\frac{d}{dt} I \approx 0$

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYASKY INTEGRAL

Introducing the L'SKY integral

$$I \sim -\int dr\, r^2 \langle \tilde{V} \cdot \tilde{V}' \rangle$$

→ Claim this is a constant for decaying isotropic turbulence

L'SKY: Integrate the Karman – Howarth Eqn.

⇓

$$\partial_t Q = 2(r\partial_r + 5)T + 2\nu\left(\partial_r^2 + \frac{4}{r}\partial_r\right)Q$$

where $Q \sim \langle \tilde{v}\tilde{v}' \rangle$, $T \sim \langle \tilde{v}\tilde{v}\tilde{v}' \rangle$

$$\int dV\,[\text{K-H eqn}] \rightarrow \partial_t\left[\int dr\, r^2 \langle \tilde{v}\tilde{v}' \rangle\right] = \partial_t I = 0$$

- Why is this result so attractive?
  - LANDAU came up with the same result via different (better) arguments
  - Used in derivation of K41 energy decay law (which is widely supported by expt.)

LANDAU: For uncorrelated vortices

$$\frac{d}{dt}\langle L^2 \rangle = \frac{d}{dt} I = \mathcal{O}(l_0/R) + \ldots \approx 0$$

L = Ang. Mom., l_0 = Vortex size, R = patch size

KOLMOGOROV: $\frac{d}{dt}\epsilon = \frac{-\epsilon^{3/2}}{l_0}$

$\epsilon \sim t^{\alpha}$, $l_0 \sim t^{\beta}$? $l_0 = l_0(\epsilon)$?

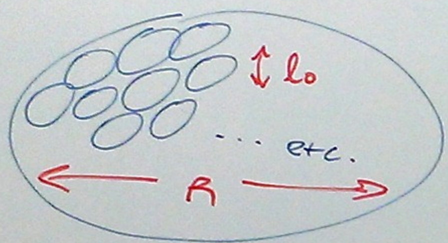
$\frac{d}{dt} I = 0 \rightarrow l_0 \sim \epsilon^{1/5} \ldots \alpha = -\frac{10}{7}$, $\beta = \frac{2}{7}$

- Why might $I = \text{const}$ be flawed?
  - Assumes no long-range correlations i.e. $\langle \tilde{v}\tilde{v}' \rangle \longrightarrow 0$ exponentially as $r \rightarrow \infty$

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYANSKY INTEGRAL

A BIT MORE DETAIL ON LANDAU'S ARGUMENTS:



3D Turb Patch of extent R
containing N identical vortices
of extent l_0

- Central limit theorem:

$$\langle L^2 \rangle = \underbrace{N \langle L_0^2 \rangle}_{\text{const}} \sim \underbrace{R^3 \rho_0^2}_{\text{const}}\ \underbrace{l_0^5 v_0^2}_{\text{const}}$$

const → const → const

+ non-interacting

→ permanence of big eddies!

$l_0 \sim \epsilon^{1/5}$

Used in K41 decay law!

- How to approach problem?
- Look for conserved quantities
- Mean angular momentum $\langle L \rangle \approx 0$

For homogeneous isotropic turbulence

- How about $\langle L^2 \rangle$?

Big questionable assumption:

Uncorrelated (non-interacting) vortices

$$\rightarrow \frac{d}{dt}\langle L^2 \rangle = \mathcal{O}(l_0/R) + \dots$$

More reasonable assumption: $l_0/R \ll 1$

→ Only surface vortices feel boundary effects

$$\frac{d}{dt}\langle L^2 \rangle \approx 0$$

L'sky Integral $I \sim \int dr\, r^2 \langle \tilde{v} \cdot \tilde{v} \rangle$

$\sim \langle L^2 \rangle$

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYASKY INTEGRAL

L'SKY & LANDAU CALLED INTO QUESTION

Assumption of zero long-range correlations suspect

↳ Largely based on intuition

Vortices correlated by pressure field (NON-LOCAL!)

$$\vec{\nabla}\cdot\left(\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}P/\rho + \nu\nabla^2\vec{v}\right) \longrightarrow \nabla^2 P = -\rho\vec{\nabla}\cdot\left((\vec{v}\cdot\vec{\nabla})\vec{v}\right) = -\rho\,\partial_{x_i}\partial_{x_j}(\tilde{v}_i\,\tilde{v}_j)$$

(switching to index notation)

* Vortex velocity field acts as a source for pressure field

$$p(x) = \frac{\rho}{4\pi}\int \frac{dx'}{|x'-x|}\,\partial_{x_i'}\partial_{x_j'}(\tilde{v}_i'\tilde{v}_j')$$

$\longrightarrow p \sim r^{-3}$ as $r\to\infty$

P+R 54: Use quasi-normal closure scheme

$$\longrightarrow \langle \tilde{v}\tilde{v}\tilde{v}'\rangle \underset{r\to\infty}{\sim} r^{-4}$$

B+P 56: Anisotropic turb

No QN approx

$\rightarrow \langle \tilde{v}\tilde{v}\tilde{p}'\rangle \sim r^{-3}$, $\langle \tilde{v}\tilde{v}\tilde{v}'\rangle \sim r^{-4}$

and $\langle \tilde{v}\tilde{v}'\rangle \sim r^{-5}$ (as $r\to\infty$) ... Nothing conclusive about isotropic turb though

SAFFMAN 66: For IC $E(k) \sim C k^2 + \mathcal{O}(k^4) + \ldots$

$\langle \tilde{v}\tilde{v}'\rangle \sim r^{-3}$, I diverges, but $\frac{d}{dt} C = 0$

OVERVIEW OF SPREADING + NON-LOCALITY

THE LOITSYASKY INTEGRAL

ASIDE ON CORRELATORS, CLOSURE, QUASI-NORMAL APPROX

Some quick notes on notation:

Plenty of diverse & confusing correlator notation ...

$R_{ij}(\vec{r}) = \langle \tilde{V}_i(\vec{x})\, \tilde{V}_j(\vec{x}+\vec{r}) \rangle$ 2nd order velocity correlator

$R_{ijk}(\vec{r},\vec{r}\,') = \langle \tilde{V}_i(\vec{x})\, \tilde{V}_j(\vec{x}+\vec{r})\, \tilde{V}_k(\vec{x}+\vec{r}\,') \rangle$ 3rd order "

$P_i(\vec{r}) = \frac{1}{\rho} \langle \tilde{P}(\vec{x})\, \tilde{V}_i(\vec{x}+\vec{r}) \rangle$ 2nd order pressure correlator

$P_{ij}(\vec{r},\vec{r}\,') = \frac{1}{\rho} \langle \tilde{P}(\vec{x})\, \tilde{V}_i(\vec{x}+\vec{r})\, \tilde{V}_j(\vec{x}+\vec{r}\,') \rangle$ 3rd order "

- $R_{\sim}$, $P_{\sim}$ above is the notation of P+R 54, B+P 56, Saff. 66
- Davidson uses "S" for R
- Chandra uses "Q" + "T" for R_{ij} + R_{ijk}
- Short hand: $\tilde{V}_i(\vec{x}) \to \tilde{V}_i$, $\tilde{V}_j(\vec{x}+\vec{r}) \to \tilde{V}_j{}'$, $\tilde{V}_k(\vec{x}+\vec{r}\,') \to \tilde{V}_k{}''$

↳ Isotropic, homogeneous turb

$\longrightarrow$ R's + P's + Q's + T's + whatever invt. under rotation + translation

$\Downarrow$

$Q_{ij}(\vec{x}) = Q_1(r)\, x_i x_j + Q_2(r)\, \delta_{ij}$

$T_{ijk}(\vec{x}) = T_1(r)\, x_i x_j x_k + T_2(r)\left(x_i \delta_{jk} + x_j \delta_{ik}\right) + T_3(r)\, x_k \delta_{ij}$

Impose $\partial_{x_i} V_i = 0$

$\longrightarrow$ relation between Q_1 + Q_2

$\therefore$ Q_{ij} can be expressed as scalar $Q(r)$

... simplify $\langle \tilde{V}_i \tilde{V}_j{}' \rangle \to \langle \tilde{V} \tilde{V}' \rangle$

Similar (more complicated) procedure for $T_{ijk} \longrightarrow$ scalar $T(r)$

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYASKY INTEGRAL

ASIDE ON CORRELATORS, CLOSURE, QUASI-NORMAL APPROX

Why all this business about correlators?

* $\partial_t \langle \hat{v}_i \tilde{v}_j' \rangle = \hat{O}_2 \{ \langle (\partial_t \tilde{v}_i) \tilde{v}_j' \rangle \}$

where $\hat{O}_2 \{ C_{ij}(r) \} = C_{ij}(r) + C_{ji}(r')$

plug in NSE $\partial_t \tilde{v}_i = -\partial_{x_k} (\tilde{v}_i \tilde{v}_k) - \frac{1}{\rho} \partial_{x_i} P + \nu \nabla^2 \tilde{v}_i$

$\longrightarrow \partial_t \langle \tilde{v}_i \tilde{v}_j' \rangle = \hat{O}_2 \{ \partial_{r'} \langle \tilde{v}_i \tilde{v}_j \tilde{v}_k' \rangle + \partial_r \frac{1}{\rho} \langle p \tilde{v}_i' \rangle + \nu \partial_{r'}^2 \langle \tilde{v}_i \tilde{v}_j' \rangle \}$

Dynamical evolution of R_{ij} dep. on triple corr. R_{ijk} + press. corr. P_i

* What if $R_{ijk} = R_{ijk}(t)$?

Similar analysis $\rightarrow \partial_t R_{ijk} = \hat{O}_3 \{ \cdots \langle \tilde{v}_i \tilde{v}_l \tilde{v}_j' \tilde{v}_k'' \rangle + \frac{1}{\rho} \langle \hat{p} \tilde{v}_i \tilde{v}_j' \rangle \cdots \}$

Never-ending hierarchy of correlators!

$\longrightarrow$ CLOSURE: End all of this madness by terminating the hierarchy at some point

Quasi-Normal (QN) approx: velocity correlations obey gaussian statistics such that 4th-order correlator boils down to a few 2nd order ones (see above)

* 4th-order Joint Cumulant

$[\hat{v}_i, \hat{v}_l, \tilde{v}_j', \hat{v}_k''] = \langle \hat{v}_i \tilde{v}_l \tilde{v}_j' \tilde{v}_k'' \rangle + \ldots$

$\ldots + \langle \hat{v}_i \hat{v}_l \rangle \langle \tilde{v}_j' \tilde{v}_k'' \rangle + \langle \hat{v}_i \tilde{v}_j' \rangle \langle \hat{v}_l \tilde{v}_k'' \rangle + \ldots$

$\ldots + \langle \hat{v}_i \tilde{v}_k'' \rangle \langle \tilde{v}_l \tilde{v}_j' \rangle$

QN approx: $[\hat{v}_i, \tilde{v}_l, \tilde{v}_j', \tilde{v}_k''] = 0$

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYASKY INTEGRAL

ENERGY SPECTRA

Typically, for small K, $E(k) \sim I K^4 + \ldots$ → L'SKY $\frac{d}{dt} I = 0$ IMPLIES PERMANENCE OF BIG EDDIES!

→ contrary to intuition...

P & R 54 : $E(k) \sim k^4$

Not permanent though... $\frac{d^2}{dt^2} I \sim \int_0^\infty dk\, E^2(k) / k^2$

(based on isotropic QN closure)

* PROBLEM: sometimes QN scheme gives $E(k) < 0$!

SAFF 66 : $E(k) \sim C k^2$

$C \sim \int dr \langle \tilde{v} \tilde{v}' \rangle$, $\frac{d}{dt} C = 0$

→ permanent, but different spectrum

DAVIDSON 2000 : Above arguments based on assumptions about:

- * initial conditions
- * velocity field statistics (correlations)

→ $\tilde{v}$ statistics could evolve during development of turbulence...

→ "FULLY-DEVELOPED TURB" has forgotten about I C's & developed full range of length scales between l_0 + l_d

ASYMPTOTIC STATE

where $\langle \tilde{v} \tilde{v}' \rangle$ → gaussian as $r \to \infty$

→ $\frac{d^2}{dt^2} I \sim \alpha v^4 l_0^3$ (recall $I \sim v^2 l_0^5$)

coupled to $\frac{d}{dt} \epsilon \sim -\epsilon^{3/2} / l_0$

* Integrate these eqns, fit free param α to experimental / numerical data

→ $\alpha \sim 0.01$ SMALL!

OVERVIEW OF SPREADING & NON-LOCALITY

THE LOITSYASKY INTEGRAL

⇩

SUMMARY

* Much debate: How to handle long-range correlations? (NON-LOCAL)
* Theory: expect these
  (to some extent)
* Experiment: agree closely with Loitsyansky & Landau (LOCAL)

→ Likely that non-local correlations exist
  they are just **weak**

→ L & L $\frac{d}{dt} I = 0$: simple physical arguments
  with predictive power

⇩

Important to
understand limitations!

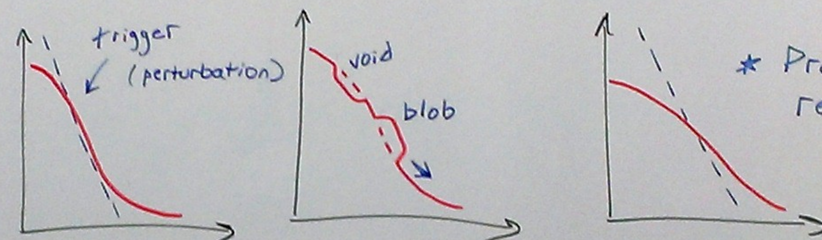
OVERVIEW OF SPREADING & NON-LOCALITY

AVALANCHES

AVALANCHING: "Spreading / Transport by gradient perturbation"

"Bursty event of correlated transport / overturning"

* Exact same concept as snow avalanche or rockslide →



* Profile relaxation event

\- - - - : critical gradient

* Buzz word "NON-LOCAL" compare/contrast:

"Self-Organized Criticality"

LOCAL

Diffusion / Fick's Law

$$\partial_t n(\vec{x},t) = -\vec{\nabla}\cdot\vec{J}(\vec{x},t)$$

$$\boxed{\vec{J}(\vec{x},t) = D\vec{\nabla} n(\vec{x},t)}$$

$$\rightarrow \partial_t n(\vec{x},t) = -D\nabla^2 n(\vec{x},t)$$

Speed limit: $\Delta x \sim \sqrt{\Delta t}$

NON-LOCAL

$$\vec{J}(\vec{x},t) \sim \int d\vec{x}'\, \underbrace{K(\vec{x},\vec{x}')}_{\text{"Kernel"}}\, \vec{\nabla} n(\vec{x}',t)$$

Super-diffusive

Ballistic propagation

$$\Delta x \sim \Delta t$$

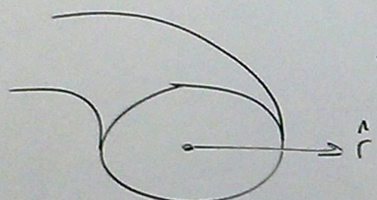
OVERVIEW OF SPREADING + NON-LOCALITY

AVALANCHES

Important / Interesting topic in + of itself

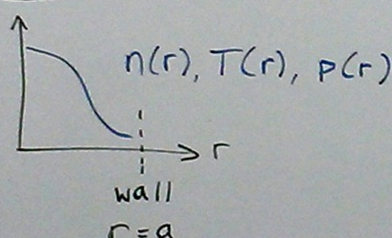
Relevant in many different fields of science

But let's talk about TOKAMAKS



* Consider 1-D
  radial confinement / transport

* Steep
  Gradients
  $\nabla n, \nabla T, \nabla P$
  → Avalanching



* Bohm diffusion : $D_B \sim T/B$
  → BAD! Fusion won't work

* Gyro-Bohm : $D = \rho_* D_B$

$$\rho_* = \rho_i / a = \frac{\text{Step size (gyroradius)}}{\text{System size (wall radius)}}$$

Fusion might work ...

Make "a" really big (ITER, \$\$!)

Gyro-Bohm Breaking: $D \neq \rho_* D_B$

→ something super-diffusive (non-local) going on ...

Quite possibly AVALANCHES!

(Difficult to see in experiments)

* Experimental Reality:

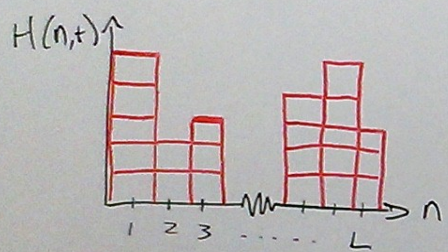
$$D \sim \rho_*^{\alpha} D_B \quad , \quad 0 \ll \alpha < 1$$

OVERVIEW OF SPREADING + NON-LOCALITY

AVALANCHES

Sandpiles + Self-Organized Criticality (SOC)

Following Hwa + Kardar 1992



Rules: IF $H(n\pm1,t) - H(n,t) > \Delta$:

$\longrightarrow H(n,t+1) = H(n,t) - N_f$

$H(n\pm1,t+1) = H(n\pm1,t) + N_f$

* BC's: Closed at $n=0$
  open at $n=L$

* Δ + N_f adjustable params
  However, scaling behavior of system is independent of these values

Drive system by depositing sand grains

$\longrightarrow$ Trigger avalanche process

$\longrightarrow$ Steady-State: SOC profile $H_{crit}(n)$

SOC:

✓ Weak external perturbation $\longrightarrow$ deposit grains

✓ Dynamical evolution $\longrightarrow$ Avalanche

✓ Eventual power-law response $\longrightarrow$ observed in size + duration of simulated sandpile avalanche events

OVERVIEW OF SPREADING & NON-LOCALITY

AVALANCHES

Sandpiles & Self-Organized Criticality (SOC)

Short timescales, distances $\longrightarrow$ Series of single isolated avalanche events
Intuitively familiar (think snow, rocks...)

Intermediate " $\longrightarrow$ Overlapping / correlated Avalanches

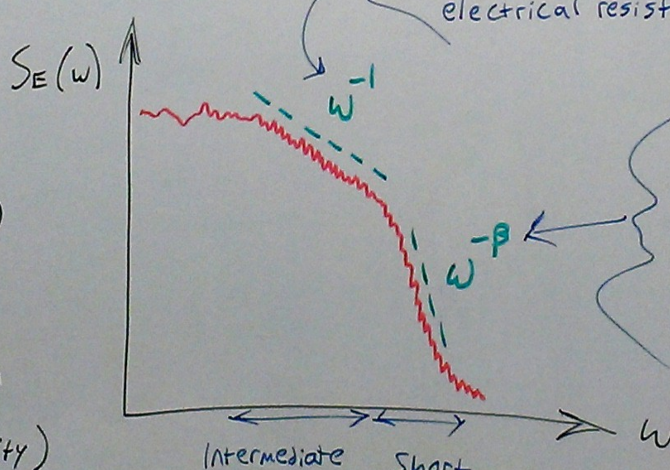
Familiar "1/F noise" seen in many physical systems
especially transport systems like traffic,
electrical resistance, earthquakes, river flow, etc.

Simulated Avalanche power spectrum:

$$S_E(\omega) = \int dt \int d\tau \, e^{-i\omega\tau} E(t) E(t+\tau)$$

where $E(t)$ = instantaneous energy dissipation

i.e. $E \sim \sum_{n=1}^{L}$ (transport activity)



For observation time
~ avalanche duration, single
avlch. signals superpose
& we get typical SOC
power law result

OVERVIEW OF SPREADING + NON-LOCALITY

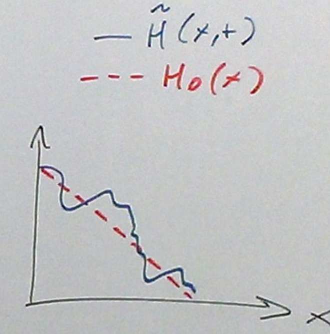
AVALANCHES

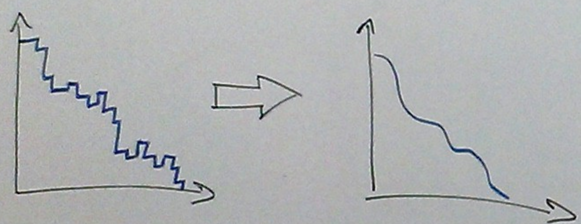
Sandpiles + Self-Organized Criticality (SOC)

* Low-Freq ω^{-1} part of spectrum dominates transport

→ suggests a "Coarse-graining" continuum approach to sandpile models

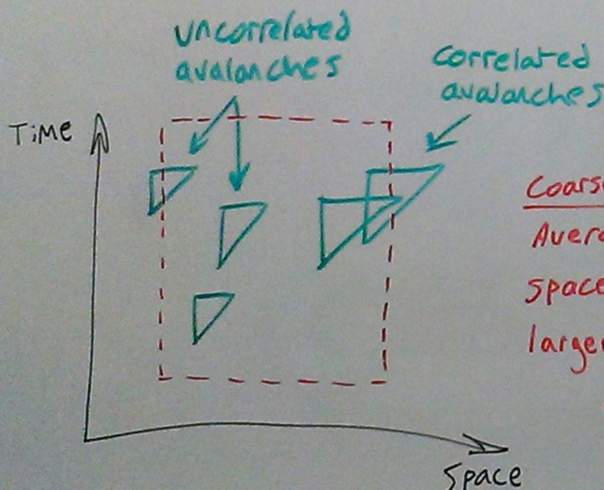
→ Hydro model:





H_0 : Steady-State self-organized profile

$h(x,t) = \tilde{H}(x,t) - H_0(x)$: Dynamical profile featuring blobs + voids



* Free $h(x,t)$

$$\int h\,dx = \text{Const} \rightarrow \partial_t h + \partial_x J_h = 0$$

* Driven: add source term " $= \eta(x,t)$

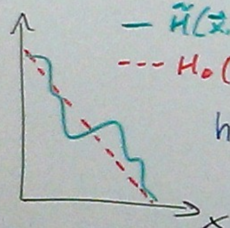
Coarse-graining: Average over space-time window larger than typical avalanche scale

OVERVIEW OF SPREADING + NON-LOCALITY

AVALANCHES

Sandpiles + Self-Organized Criticality (SOC)

* More on hydro sandpile model



— $\tilde{H}(\vec{x},t)$

--- $H_0(x) = \lambda_0 (L - x)$

$h(\vec{x},t) = \tilde{H} - H_0$

$$\partial_t h(\vec{x},t) + \vec{\nabla} \cdot \vec{J}(h) = \eta(\vec{x},t)$$

where $\vec{\nabla} = \nabla_{\parallel} \hat{x}_{\parallel} + \nabla_{\perp} \hat{x}_{\perp}$, $\hat{x}_{\parallel}$ directed along gradient (slope λ_0)

* Need to calc. $\vec{J}(h)$

→ complicated in general ... Use symmetry arguments

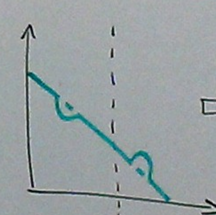
Symmetries of problem:

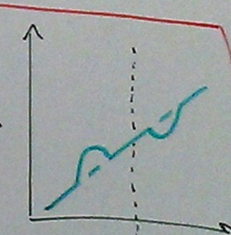
$$\longrightarrow \vec{J}(h) = \underbrace{-\nu_{\perp} \vec{\nabla}_{\perp} h - \nu_{\parallel} \vec{\nabla}_{\parallel} h}_{\nu_{\perp,\parallel} \sim \text{ surface tension}} + \underbrace{\frac{\lambda_0}{2} h^2 \hat{x}_{\parallel}}_{\text{due to gradient}}$$

Rotation	Translation	Reflection
✓ $x_{\perp}$	✓ $x_{\perp}$	Not $x_{\parallel}$ or h
	✓ $x_{\parallel}$	BUT →
Not $x_{\parallel}$	Not h (debatable...)	

JOINT REFLECTION SYMMETRY

$x_{\parallel} \to -x_{\parallel}$, $h \to -h$





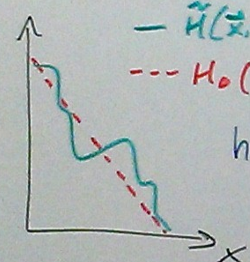
i.e. Change gradient direction, swap blobs + voids

OVERVIEW OF SPREADING & NON-LOCALITY

AVALANCHES

Sandpiles & Self-Organized Criticality (SOC)

* More on hydro sandpile model



— $\tilde{H}(\vec{x},t)$

--- $H_0(x) = \lambda_0 (L - x)$

$h(\vec{x},t) = \tilde{H} - H_0$

$$\boxed{\partial_t h - (\nu_{\parallel} \nabla_{\parallel}^2 + \nu_{\perp} \nabla_{\perp}^2) h + \frac{\lambda_0}{2} \nabla_{\parallel} h^2 = \eta}$$

Non-linearity   source

* Smells like

BURGERS EQN:

$$\boxed{\partial_t \xi + \xi \partial_x \xi - \nu \partial_x^2 \xi = \eta}$$

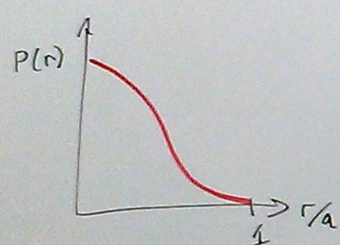
→ Exact solutions for $\eta = 0$

Ballistically propagating Shock Fronts

OVERVIEW OF SPREADING + NON-LOCALITY

AVALANCHES

NOW RELATE SANDPILE TO TOKAMAK GRADIENTS



... could also talk about $T(r)$, $n(r)$

Micro-scale: sand grain size $\longleftrightarrow$ gyro rad. ρ_i ... single avalanches

Meso-scale: overlapping avalanches $\longleftrightarrow$ "streamers"? = coherent convective cell

1/f noise : $|\tilde{\Gamma}_\omega|^2 \sim \omega^{-1}$

Prob. of event size l : $P(l) \sim l^{-1}$

Macro-Scale: L_P, L_T press + temp scale lengths $\sim$ system size "a"

Just like in sandpiles, constrain structure of avalanche flux $\Gamma(\delta P)$ ($\longleftrightarrow$ $J(h)$ in sandpile) using JOINT REFLECTION SYMMETRY

$$\partial_t \delta P + \partial_x \Gamma(\delta P) = \tilde{S}$$

$$\text{JRS: } \Gamma(\delta P) \sim \frac{\lambda}{2}\delta P^2 - D_0 \partial_x \delta P$$

$$\longrightarrow \partial_t \delta P + \lambda \delta P \partial_x \delta P - D_0 \partial_x^2 \delta P = \tilde{S}$$

Noisy / Driven Burgers Eqn!

δP = perturbation in press. profile

$\longrightarrow$ Ballistic Shock Fronts

OVERVIEW OF SPREADING + NON-LOCALITY

AVALANCHES

SUMMARY

* Experimental evidence:
  - Gyro-Bohm breaking
  - Non-Local Phenomena
* Sandpile models
  - origin of "1/F noise"
  - Ballistic transport via Burgers
* ∇P, ∇T, ∇n analogous to sandpiles

  Avalanching → Likely suspect for Non-Local transport

* Difficult to observe single avalanche experimentally ...

  → too fast!

  Usually just see 1/F transport spectrum

  → Need FAST DIAGNOSTICS

  ... How Fast?

* In order to "see" avalanches we need $\rho_i/L \ll 1$ ... i.e. we need a machine w/ small dimensions + high field

	LAPD (UCLA)	Surko Magnet (UCSD)	Alcator (MIT)
geometry	cylindrical	cylindrical	toroidal
B (G)	2 500	50 000	80 000
L (cm)	37	7.5	22
ρ_i (cm)	9×10^{-2}	4.6×10^{-3}	2.85×10^{-3}
ρ_i/L	2.4×10^{-3}	6.1×10^{-4}	1.3×10^{-4}

OVERVIEW OF SPREADING & NON-LOCALITY

K – ε MODELS

INTRO

* K-ε: Spreading due to NonLinear coupling

* Essentially a 2-equation Closure scheme for computing Reynolds Stress $\langle \tilde{v}_i \tilde{v}_j \rangle$

* Most widely used + successful numerical turbulence model

  → Computational Fluid Dynamics

  HUGE Industry: Mech. Engineering
  Aerospace "
  Chemical "
  Bio "
  Etc.

* Slightly less tracable from a theoretical point of view...

CLOSURE: Modelling requires small-scale cutoff to be practical

→ "unresolved scale"

→ "Sub-grid Scale Models"

DIVERSITY: Numerous variants of K-ε, i.e. many different ways to express the 2 closure eqns

→ Different variants better/worse for different flows

→ Often include adjustable params which must be fit to experiment

OVERVIEW OF SPREADING & NON-LOCALITY

K – ε MODELS

* Start from the beginning

- Reynolds-Avg'd Navier Stokes: NSE, $v_i = \langle v_i \rangle + \tilde{v}_i$

  (RANS) $\longrightarrow \frac{D}{Dt} \langle v_j \rangle = \nu \nabla^2 \langle v_j \rangle - \partial_{x_i} \langle \tilde{v}_i \tilde{v}_j \rangle - \frac{1}{\rho} \partial_{x_j} \langle p \rangle$

- Get pressure from Poisson eqn

  $\vec{\nabla} \cdot \overrightarrow{(\text{RANS})} \longrightarrow \frac{1}{\rho} \nabla^2 \langle p \rangle = \partial_{x_j} \langle v_i \rangle \partial_{x_i} \langle v_j \rangle + \partial_{x_i} \partial_{x_j} \langle \tilde{v}_i \tilde{v}_j \rangle$

- Central to any turbulence model: how to calc "unknown" Reynolds Stress $\langle \tilde{v}_i \tilde{v}_j \rangle$?

  $\longrightarrow$ turbulent viscosity $\langle \tilde{v}_i \tilde{v}_j \rangle \sim K \delta_{ij} - \nu_T \left(\partial_{x_i} \langle v_j \rangle + \partial_{x_j} \langle v_i \rangle \right)$

- Okay ... then how to calc ν_T ?

  $\longrightarrow$ Dimensionally, need length & velocity scale: $\nu_T \sim V_* l_*$

- Mixing length approach: $\nu_T = l_*(\vec{x},t) \, |\partial_j \langle v_i \rangle|$

  $\longrightarrow$ Incomplete, need to specify l_*

OVERVIEW OF SPREADING & NON-LOCALITY

K – ε MODELS

- Recap: RANS, $\langle \tilde{v}_i \tilde{v}_j \rangle \to \nu_{turb} \sim v_* l_*$
- One-equation model:

$v_* \sim K^{1/2}$ (k = local turb energy density)

Introduce transport equation for K

Turb energy production

$$\partial_t K + (\langle \vec{v} \rangle \cdot \vec{\nabla}) K = \vec{\nabla} \cdot \Big(\underbrace{\frac{\nu_T}{\sigma_K}}\, \vec{\nabla} K \Big) + \mathcal{P} - \varepsilon$$

$$\nu_T \sim K^{1/2} l_* \longrightarrow \text{NL!} \qquad \varepsilon \sim K^{3/2} / l_*$$

$\longrightarrow$ 1-eqn does better than mixing length approach but still incomplete, need to specify l_*

OVERVIEW OF SPREADING & NON-LOCALITY

K − ε MODELS

* Recap: RANS, $\langle \tilde{v}_i \tilde{v}_j \rangle \rightarrow \nu_{turb} \sim v_* l_*$

* 2-eqn model : implement dynamical eqns for K & ε

$K \rightarrow$ energy density

$\varepsilon \rightarrow$ energy dissipation

Good thing about these two sub-grid quantities : we can form

<u>Length scale</u> : $l_* \sim K^{3/2}/\varepsilon$

<u>Time scale</u> : $\tau \sim K/\varepsilon$

<u>Viscosity</u> : $\nu_T \sim K^2/\varepsilon$

<u>K−ε Model constants</u> (5 of them)

ε-eqn : $C_{\varepsilon 1}, C_{\varepsilon 2}, \sigma_\varepsilon$

K-eqn : σ_K

$\nu_T = C_\mu K^2/\varepsilon$

- Need transport eqn for ε

$$\longrightarrow \boxed{\partial_t \varepsilon + \langle \vec{v} \rangle \cdot \vec{\nabla} \varepsilon = \vec{\nabla} \cdot \left(\frac{\nu_T}{\sigma_\varepsilon} \vec{\nabla} \varepsilon \right) + C_{\varepsilon 1} \frac{P \varepsilon}{K} - C_{\varepsilon 2} \frac{\varepsilon^2}{K}}$$

- Recall transport eqn for K

$$\boxed{\partial_t K + \langle \vec{v} \rangle \cdot \vec{\nabla} K = \vec{\nabla} \cdot \left(\frac{\nu_T}{\sigma_K} \vec{\nabla} K \right) + P - \varepsilon}$$

* FULL-BLOWN K−ε MODEL!

$\longrightarrow$ <u>CLOSED</u>, no free param l_*

OVERVIEW OF SPREADING & NON-LOCALITY

K – ε MODELS

* K – ε odds & ends

SPECIAL CASES

- Homogeneous turbulence

  $\rightarrow \partial_{x_i} = 0$

$$\begin{cases} \frac{d}{dt} K = P - \varepsilon \\ \frac{d}{dt} \varepsilon = C_{\varepsilon_1} \frac{P \varepsilon}{K} - C_{\varepsilon_2} \frac{\varepsilon^2}{K} \end{cases}$$

- Decaying turbulence

  Production $P \rightarrow 0$

$$K(t) = K_0 (t/t_0)^{-n}$$

$$\varepsilon(t) = \varepsilon_0 (t/t_0)^{-(n+1)}$$

where $t_0 = n \frac{K_0}{\varepsilon_0}$ and $n = \frac{1}{C_{\varepsilon_2} - 1}$

SHORTCOMINGS

* Flow complexity ↑ ⇒ Model accuracy ↓

* Different flows ⟷ Slightly different values for model constants C_{ε_i} etc. (for most realistic results)

  - K – ε solver comes pre-loaded w/ constant values which are a compromise between most commonly used flows (i.e. pipes, airfoils ...)

MODIFICATIONS

* Most involve altering ε-equation, like adding extra terms

* In general, specific mods. target specific flows ...

  ⟶ No clear universal improvements

OVERVIEW OF SPREADING & NON-LOCALITY

K–$\mathcal{E}$ MODELS

SUMMARY:

* sub-grid (unresolved) transport equations
  for turb. energy density $K = \langle \tilde{v}^2 \rangle$
  and " " dissipation $\mathcal{E} \sim \Delta K / \Delta t$
* Widely used in the engineering community
  → Handy numerical tool
  Difficult to extract theoretical understanding

LOOKING FORWARD:

* K–$\mathcal{E}$ transport equations look like
  Reaction-diffusion system: $\partial_t \xi = \underbrace{D \nabla^2 \xi}_{\text{Diff.}} + \underbrace{f(\xi)}_{\text{Reac.}}$
  → "Invasion Front" propagation

→ Fisher, Fitzhugh-Nagumo: Next section of notes!

OVERVIEW OF SPREADING & NON-LOCALITY

REFERENCES

LOITSYANSKY

- P. A. Davidson "Was Loitsyansky correct? A review of the arguments" Journal of Turbulence (2000) 006
- P.G. Saffman "The large-scale structure of homogeneous turbulence" Journal of Fluid Mechanics (1967) 27
- I. Proudman + W.H. Reid "On the decay of a normally distributed + homogeneous turbulent velocity field" Phil. Trans. Royal Soc. London (1954) 247
- G.K. Batchelor + I. Proudman "The large-scale structure of homogeneous turbulence" Phil. Trans. R. Soc. Lond. (1956) 248

AVALANCHES

- Hwa + Kardar "Avalanches, hydrodynamics, and discharge events in models of sandpiles" Phys. Rev. A (1992) 45

K-ε MODELS

- S. B. Pope "Turbulent Flows" Cambridge University Press (2000)

MULTIPLE SECTIONS

- E. Spiegel "The theory of turbulence; S. Chandrasekhar's 1954 Lectures" Springer (2011)
- P. Diamond, S. Itoh, K. Itoh "Modern Plasma Physics Vol 1: Physical Kinetics of turbulent plasmas" Cambridge University Press (2010)